On the Synthesis of Periodic Linkages with a Specific Constant Poisson's Ratio

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Abstract. Poisson's ratio is one of the most studied material properties that can be designed in mechanical metamaterials. However, in most studies so far, Poisson's ratio is not constant for larger compressions. Only for structures in which $\nu = -1$, structures with a constant Poisson's ratio have been demonstrated. This paper studies the design of planar mechanical metamaterials with a constant Poisson's ratio based on the pantograph, inversor, straight-line and parabolograph mechanisms. Using these classical mechanisms as building blocks, periodic mechanisms with $\nu = -1, -\frac{1}{2}, 0$ and 1 are proposed.

Keywords: mechanical metamaterials, periodic linkages, auxetics, Poisson's ratio, pantograph, inversor, straight-line mechanism, parabolograph

1 Introduction

Poisson's ratio describes the deformation of a material in the directions perpendicular to a uniaxially applied load. This ratio is the subject of a large body of research into artificially designed material structures called mechanical metamaterials [1, 2]. Especially, material structures with a negative Poisson's ratio, also named auxetics [3–5], are of interest lately, because of the counter-intuitive property that they contract in the direction perpendicular to an applied uniaxial compression. This property leads to an increased shear modulus and impact resistance of these material structures [4].

One category of auxetic structures consists of periodic linkages. These built up from unit cells consisting of rigid bars or bodies, connected by hinges [6–8]. In these structures, the periodic degrees of freedom have been designed such that when one of the dimensions of the unit cell decreases, the others also decrease and vice-versa. This effect has been demonstrated for planar periodic linkages [9–11] as well as for spatial structures [12].

Poisson's ratio is an infinitesimal property; its value is defined at a state of the structure based on small deformations around that state. In most mechanical meta-materials found in literature this value is not constant for larger deformations. A noteable exception to this are structures with a constant Poisson's ratio of -1 based on rotating squares [9] or triangles [13].

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T. Uhl (ed.), *Advances in Mechanism and Machine Science*,
Mechanisms and Machine Science 73,
https://doi.org/10.1007/978-3-030-20131-9_25

Periodic linkages could be used as a starting point for the design of elastic mechanical metamaterials. When a linkage has been designed with the desired kinematic properties, i.e. desired Poisson's ratio, then, as a next step, an elastic structure could be derived from it, obtaining a metamaterial with similar kinematic properties.

In this paper, we will show that planar periodic linkages with various constant Poisson's ratios can be created based on classical linkages, specifically, the pantograph, the inversor, the straight-line mechanism, and the parabolograph. We first determine the necessary transmission function between a vertical input and a horizontal output of a unit cell. Then, we present four periodic new linkages with constant Poisson's ratios of $-1, -\frac{1}{2}, 0$ and 1 and illustrate the connection of these periodic linkages to their classical counterparts.

2 Planar Periodic Linkages

Periodic linkages consist of a basic mechanism, the unit cell, which is copied along two non-parallel vectors to fill the whole plane [14, 15]. This is illustrated in figure 2. The unit cell of such a linkage is always a parallelogram, such that the tiling corresponds to a Bravais lattice [16].

We study a single unit cell of this structure and impose periodic boundary conditions to it to maintain connectivity between neighboring unit cells. In this paper, we only study mechanisms with a single degree of freedom, and therefore all unit cells deform in the same way. An other study [7] has investigated structures with more than one degree of freedom using Bloch-wave analysis, where the deformation patterns are still periodic, but possibly with a larger period than the constructed lattice.

3 Poisson's Ratio in Periodic Linkages

For materials, Poisson's ratio is defined by:

$$\nu_{ij} = -\frac{d\epsilon_j}{d\epsilon_i}, \tag{1}$$

where the labels i, j denote perpendicular directions and $d\epsilon_i$ is an infinitesimal strain in the ith direction [9]. For planar periodic mechanisms, we can use the width and height of the unit cell to express a similar ratio. We follow Grima and Evans (2000) [9], and express the effective Poisson's ratio in terms of these dimensions:

$$\nu_{HW} = -\frac{H}{W}\frac{\partial W}{\partial H}, \tag{2}$$

Where W and H are respectively the width and height of a unit cell. Using this equation, we can express the width of a unit cell as a function of its height:

$$W(H) = CH^{-\nu_{HW}}, \tag{3}$$

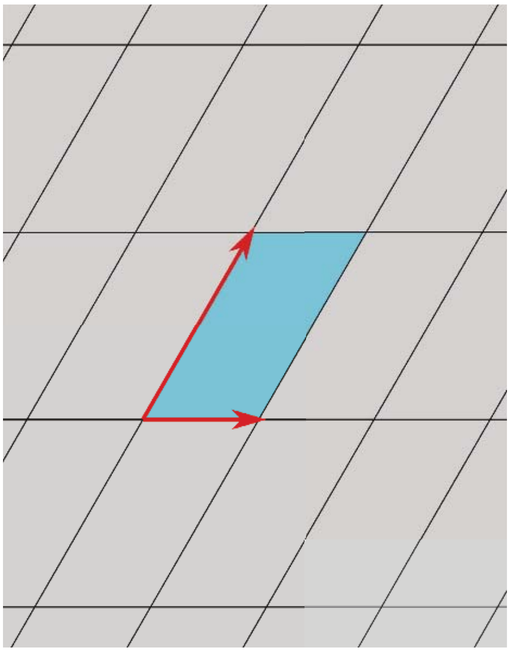

Fig. 1. A periodic linkage consists of a unit cell (highlighted in cyan) which is copied along two non-parallel vectors (red arrows) to fill the plane. Within this unit cell, a linkage is constructed with periodic boundary conditions, maintaining connectivity between neighboring unit cells.

where C is an arbitrary constant. When designing a periodic linkage with a certain Poisson's ratio, we look for a linkage with this transmission between the two perpendicular directions. A useful observation here is that $\nu_{WH} = (\nu_{HW})^{-1}$, so that when a periodic linkage is obtained for a specific Poisson's ratio, we automatically also obtain a linkage for the inverse Poisson's ratio by rotating the linkage by 90 degrees.

In the following, we will show how periodic linkages with a constant Poisson's ratio can be obtained from well-known planar linkages for the cases of $\nu = -1, -\frac{1}{2}, 0$ and 1.

3.1 $\nu = -1$: Pantograph linkages

In the field of auxetics, the case of $\nu = -1$ has been widely studied because this is the limiting case for isotropic materials due to thermodynamic considerations. For these structures, the change in the width of the structure is directly proportional to the change in the height of the structure:

$$W(H) = CH, \tag{4}$$

Where C depends on the geometry of the structure.

The shape of the unit cell does not change, but it does change in size. Therefore, the degree of freedom of such a linkage corresponds to dilation. One of the best known linkages that has this property is the pantograph [17]. Four pantograph linkages can be coupled together to form a unit cell of the structure, as is shown in figure 2. This structure has been described previously by Attard et al. [11]. For the drawn structure, $C = \frac{a}{b}$.

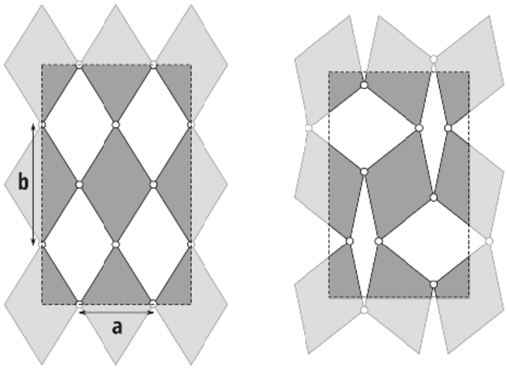

Fig. 2. Four pantographs can be combined to form a unit cell with $\nu = -1$. The grey plates are rigid and allowed to hinge at their corners. The unit cell is indicated by a dashed line. This structure is adapted from [11].

The structure in Fig. 2 can be generalized further to the pantographs presented in Fig. 3. Here, we take a parallelogram and construct a pantograph from it by following the construction described in [17]. This pantograph is then mirrored along both edges of the parallelogram to obtain a unit cell consisting of four copies of the pantograph. The rigid triangles of these pantographs are coupled to their neighbours such that they form rigid quadrilaterals, each of them shared between four separate pantograph mechanisms. For this structure, $C = \frac{b}{a} \sin \alpha$.

3.2 $\nu = 1$: Inversor linkages

For Poisson's ratio 1, the unit cell must have a constant area for the full range of motion. Then, the width of the structure can be written as:

$$W(H) = \frac{C}{H}. \tag{5}$$

This means that the width of the structure is inversely proportional to its height,. This behavior can be achieved by using an inversor linkage [18]. Figure 4 presents a design of such a periodic linkage that was derived from Fig. 49 of Artobolevskii's book [18]. For this structure, $C = 4a^2 - b^2$. When the rigid squares rotate, the sliders on the edges of the indicated unit cell move such that the area of the unit cell remains constant. Therefore, the Poisson's ratio of this linkage is 1.

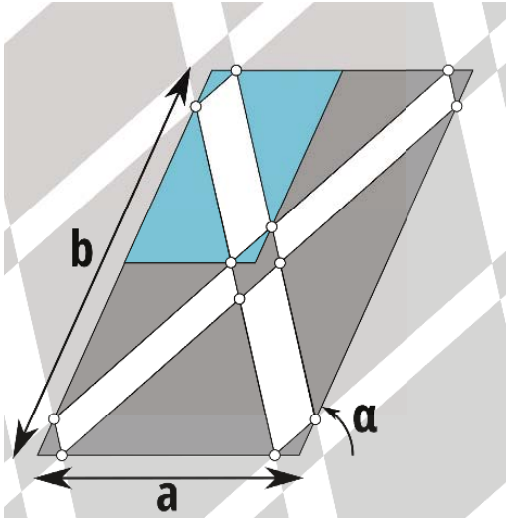

Fig. 3. Proposed design of a planar parallelogram-shaped unit cell, we can construct a set of four pantograph linkages, merged together such that there is one degree of freedom for the mechanism which corresponds to $\nu = -1$. The gray faces in this drawing are rigid elements and connected by revolute joints at their corners. The pantograph in the upper left corner is highlighted.

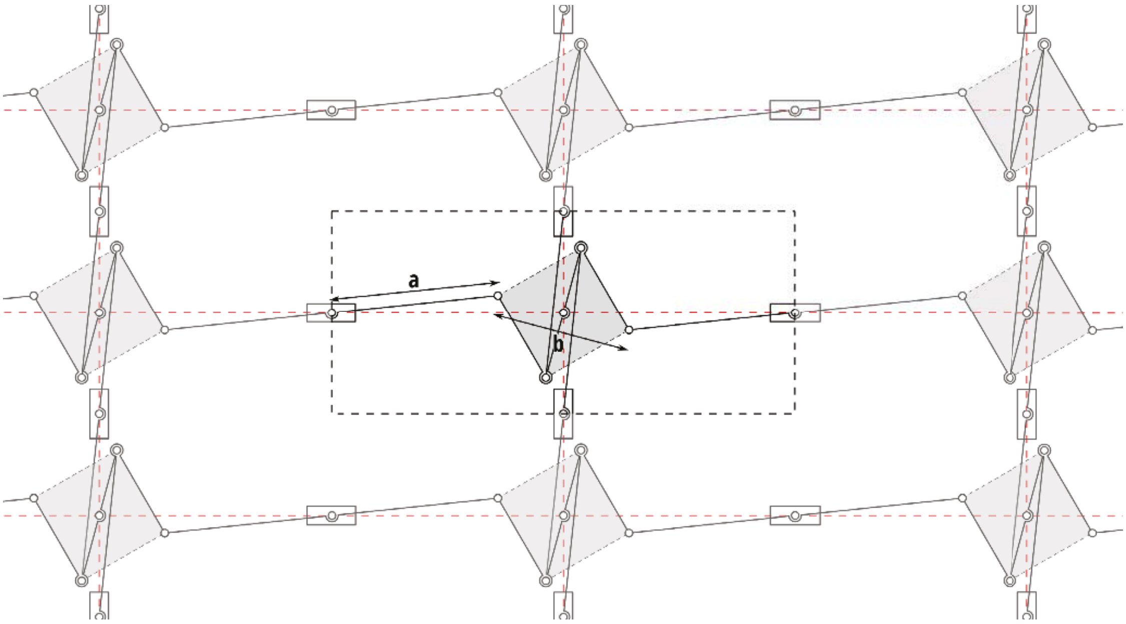

Fig. 4. Proposed design of a periodic linkage with a Poisson's ratio of 1 based on inversor linkages. Red dashed lines indicate the slider paths. The central unit cell of the linkage is indicated with a dashed black line. The gray squares indicate rigid bodies which have a pivot at the intersection of the horizontal and vertical slider paths. The diagonals of the grey squares have length b and the four bars have length a.

3.3 $\nu = 0$: Straight-line mechanisms

For a Poisson's ratio of zero, there should be no dependence between the width and height of the unit cell. This can be formulated as:

$$W(H) = C. \tag{6}$$

For 1-DoF linkages this property is found in straight-line mechanisms. In principle, any straight line mechanisms could be used to achieve a periodic $\nu = 0$ linkage. In Fig. 5, we present a periodic linkage that is based on Hart's A frame. Hart's A-frame has two stationary points at the base of the A, and draws a straight line, perpendicular to the base, with its top point. We couple two of these linkages together, one rotated by 180 degrees, to form a parallelogram-shaped unit cell. When it is actuated, the corners of the unit cell will move along straight, parallel lines, thus changing the height of the unit cell while preserving its width. For the drawn structure, $C = a$.

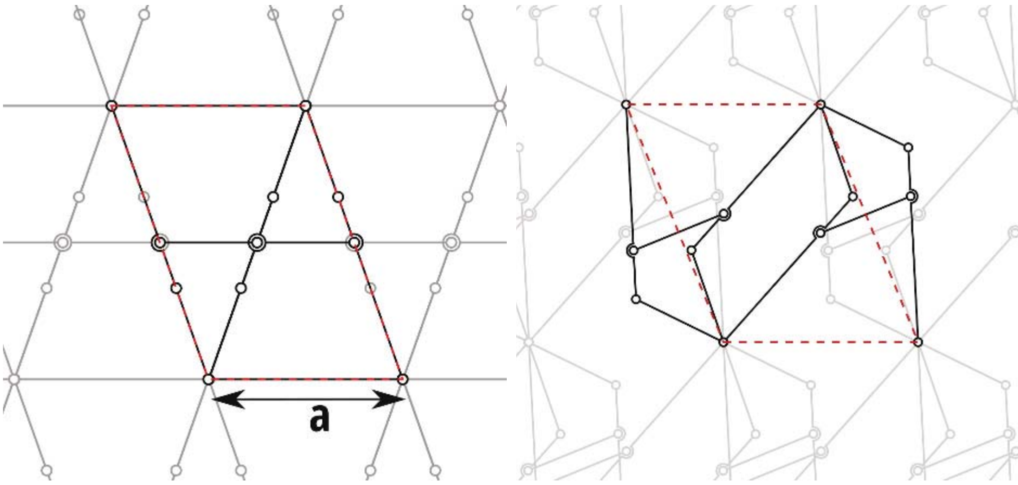

Fig. 5. Proposed design of a periodic linkage with $\nu = 0$, based on Hart's A-frame. Shown are two configurations within its range of motion. The unit cell is indicated by the red dotted line and drawn with black lines, The rest of the lattice is drawn in gray. When this linkage moves, the height of the unit cell decreases, but its width remains the constant distance a.

3.4 $\nu = -\frac{1}{2}$: Parabolograph linkages

For a Poisson's ratio of $-\frac{1}{2}$ the width of the unit cell should be given by

$$W(H) = C\sqrt{H}. \tag{7}$$

To achieve such behavior, we can use linkages from the class of Parabolographs. These are planar linkages designed to draw parabolas in the plane. Here, we will use Antonov's parabolograph (see p. 132 of Artobolevskii's book [18] for more information on this mechanism).

Antonov's parabolograph consists of three moving links, connected by revolute and prismatic joints to form a one-degree of freedom mechanism. This mechanism is shown in Fig. 6. When the angular link in this mechanism rotates around point O, point A moves along the vertical guide and point B follows the parabolic curve indicated by the green dotted line.

If we take the distance AB of this mechanism to be proportional the width of the unit cell and vertical distance between points O and A to be proportional to the height of the unit cell, we obtain a unit cell with a Poisson's ratio equal to $-\frac{1}{2}$. In this case, $C = \sqrt{\frac{c}{2}}$, where c is the distance between lines a and b.

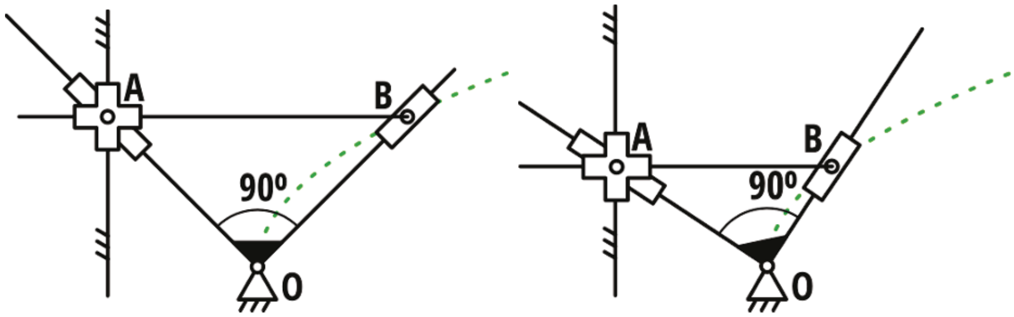

Fig. 6. Antonov's Parabolograph in two different configurations. Point O is fixed to the ground, point A is fixed to a vertical line. When actuated, Point B moves along a parabolic path, indicated by the green dotted line. Figure adapted from Fig. 215 in Artobolevskii's book [18]

We can mirror this linkage along both the vertical and horizontal axes to obtain the tileable version presented in Fig. 7. Here, the horizontal lines labeled a and b indicate guides for the sliding connections and the central unit cell is indicated by a dashed red line. This linkage has been rotated by 90 degrees with respect to Fig. 6 to achieve $\nu_{HW} = -\frac{1}{2}$.

When actuated, the four corners of the unit cell follow parabolic curves, such that $H \propto W^2$.

4 Discussion

In this paper, we have shown how classical linkages, specifically the pantograph, inversor, straight-line mechanisms and the parabolograph, can be used to create periodic linkages with various constant effective Poisson's ratios. We have presented designs for a constant Poisson's ratio of $-1, -\frac{1}{2}, 0$ and 1. We are, however not limited to these examples. The curves required to obtain these values of ν can be generated by a number of different linkages, which then can be converted into a variety of periodic counterparts.

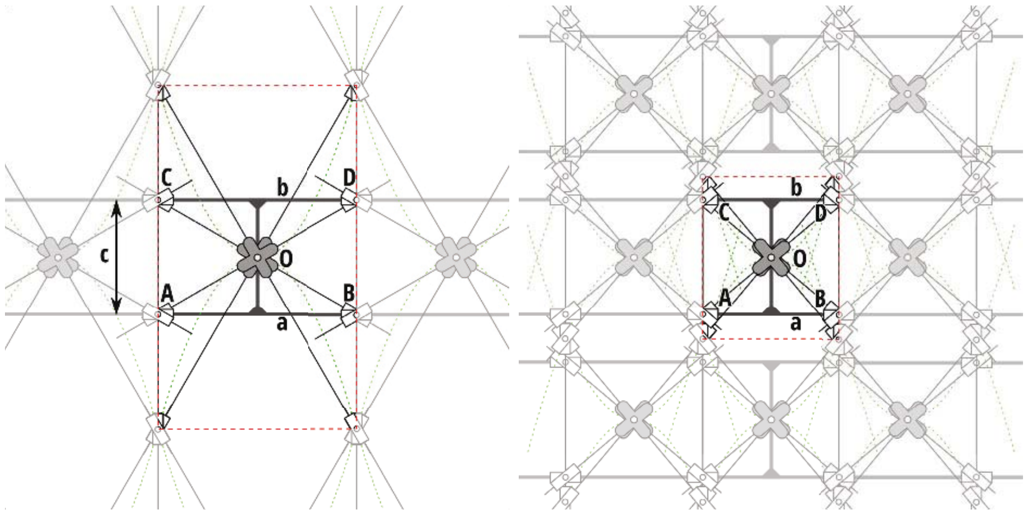

Fig. 7. Proposed design of a periodic linkage based on Antonov's Parabolograph with a constant Poisson's ratio of $-\frac{1}{2}$. Two different configurations are shown. The unit cell is indicated by a dashed red line, the corners of this unit cell follow the green dashed parabolic paths. The points A, B, C and D slide over lines a and b. Point O is situated at the center of the unit cell. The two crossed bars pivot around this point.

It is interesting to note from equation 3 that all negative integer Poisson's ratios can be achieved by linkages drawing mononomial curves, the existence of which have been proven generally by Kempe in 1875[19]. Consequently, linkages with $\nu = -\frac{1}{n}$ also exist for every positive integer n and can be obtained by rotating the linkage with $\nu = -n$ by 90 degrees. Versions of these periodic linkages with positive Poisson's ratio could be obtained by integrating an inversor within the design.

For pantograph-based linkages with $\nu = -1$, it has been shown that the kinematics can be matched by an elastic structure with a square array of round holes[20, 21]. This could significantly simplify the manufacturing for these metamaterials. Similarly, elastic structures could be designed to match the kinematics of the linkages shown in this paper. If the designs in this paper are best suitable for this and how sliders could be desgigned elastically was not yet investigated.

5 Conclusion

In this paper, we have presented 4 designs of periodic mechanisms with a constant effective Poisson's ratio of $-1, -\frac{1}{2}, 0$ and 1. These mechanisms were constructed based on the pantograph, inversor, straight-line and parabolograph linkage, respectively, which were adapted and tiled in a periodic grid. In this way, infinitely large linkages are constructed with a specified expansion or contraction in the horizontal direction as a response to a vertical actuation.

We have shown that it is possible to create periodic mechanisms for a variety of constant Poisson's ratios, where, in current literature, this has only been demonstrated for $\nu = -1$. In the discussion, we have indicated how linkages with other constant Poisson's ratios can be constructed using a similar approach to the one presented in this paper.

These mechanisms could, in future work, be implemented in monolytic, elastic structures with the specified Poisson's ratio that remains constant for large deformations. In this way, artificial materials can be constructed with predictable and tailored material properties that stay constant for large deformations.

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